Analysis of Algorithm

Extended Karatsuba, Graph

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Recap Counting Sort

Concept:

- Counts occurrences of each unique element.
- Uses this count to determine final positions in sorted order.

Time Complexity:

- Best/Average/Worst: $O(n + k)$ (where k is the range of numbers)

Stable: Yes.

Limitation: Inefficient for large ranges.

Recap Bucket Sort

Concept:

- Distributes elements into multiple "buckets".
- Each bucket is sorted individually using another sorting algorithm.

Time Complexity:

- Best/Average: $O(n + k)$ (where k is the number of buckets)
- Worst: $O(n^2)$ (if all elements land in one bucket)

Stable: Yes (if the inner sorting algorithm is stable).

Recap Radix Sort

Concept:

- Sorts numbers digit by digit, starting from the least or most significant digit.
- Uses a stable sorting algorithm (like Counting Sort) at each step.

Time Complexity:

- Best/Average/Worst: $O(d(n+k))$ (where d is the number of digits, k is the base)

Stable: Yes.

Limitation: Requires extra space and works best when the number of digits is not too large.

Extended Karatsuba

As discussed in Lecture #7, Karatsuba is an algorithm that multiplies two integers by dividing them into two parts, achieving a time complexity of $O(n^{\log_2 3}) \approx O(n^{1.585})$.

- Let two numbers:

$$P = [\underline{\qquad}A\underline{\qquad} \mid \underline{\qquad}B\underline{\qquad}]$$

$$Q = [\underline{\qquad}X\underline{\qquad} \mid \underline{\qquad}Y\underline{\qquad}]$$

  We make three recursive calls, Hence:

- **Recurrence Relation:**

$$T(n) = 3T(n/2) + O(n)$$

- **Master Theorem:**

$$T(n) = \Theta(n^{\log_2 3})$$

TOOM-3 Algorithm

Concept:

- TOOM-3 is an extension of Karatsuba's multiplication algorithm.
- It divides an integer into 3 parts and performs multiplication on them.
- The goal is to reduce recursive calls and minimize operations.

Integer Representation:

$$P = [\underline{\quad\quad} a \underline{\quad\quad} | \underline{\quad\quad} b \underline{\quad\quad} | \underline{\quad\quad} c \underline{\quad\quad}] \quad \text{(n bits)}$$

$$Q = [\underline{\quad\quad} x \underline{\quad\quad} | \underline{\quad\quad} y \underline{\quad\quad} | \underline{\quad\quad} z \underline{\quad\quad}] \quad \text{(n bits)}$$

Assumption:

- n is divisible by 3 to allow equal partitioning.

TOOM-3 Multiplication Expansion

Multiplication:

$$
\begin{aligned}
P \times Q &= (a \cdot 2^{\frac{2n}{3}} + b \cdot 2^{\frac{n}{3}} + c) \times (x \cdot 2^{\frac{2n}{3}} + y \cdot 2^{\frac{n}{3}} + z) \\
&= (ax \cdot 2^{\frac{4n}{3}} + ay \cdot 2^{n} + az \cdot 2^{\frac{2n}{3}} + \\
&\qquad bx \cdot 2^{n} + cx \cdot 2^{\frac{2n}{3}} + \\
&\quad cz + \quad bz \cdot 2^{\frac{n}{3}} + by \cdot 2^{\frac{2n}{3}} + \\
&\qquad cy \cdot 2^{\frac{n}{3}})
\end{aligned}
$$

Separation:

1. Now we have to reduce it into 5 essentials. So,
2. We just want to separate: **ay, bx** and bz,cy.
3. Terms: az,cx,by can be achieved by *essentials*, $\frac{3+4}{2}$.

TOOM-3 Essentials

Essentials:

1. ax
2. cz
3. $(a + b + c) \times (x + y + z)$

Expansion:

$$
\begin{aligned}
(a + b + c) \times (x + y + z) = (&\mathbf{ax} + \mathbf{ay} + \mathbf{az} + \\
&\mathbf{bx} + \mathbf{by} + \mathbf{bz} + \\
&\mathbf{cx} + \mathbf{cy} + \mathbf{cz})
\end{aligned}
$$

Eliminating Redundant Terms:

1. We do not need to worry about ax and cz as they are already in the essentials.
2. The required terms are the diagonal ones: **az, by, cx**.

Eliminating Redundant Terms:

1. We need to get rid of the remaining four terms: ay, bx , bz,cy.
2. To remove these, we must determine the **Essential 4**, which helps in eliminating these unwanted terms.

Essential 4: First Attempt

1. We aim to eliminate ay, bx, bz, cy by identifying common patterns.
2. Trying $(-b + c) \times (y - z)$:

$$= -by + bz + cy - cz$$

3. However, this does not include ay or bx since we have not accounted for a or x in multiplication.

Essential 4: Corrected Approach

$(a - b + c) \times (x - y + z) = ax - ay + az - bx + by - bz + cx - cy + cz$

Term Classification:

- **Positive Terms:** $ax + az + cx + cz + by$
  - ax, cz (Already part of essentials)
  - az, by, cx (Their sum will make us happy to do this direct to essential 5)
- **Negative Terms:** -ay, -bx,-bz, -cy (These were the terms targeted for cancellation in the 3rd essential)

Essential 5: Testing with Different Coefficients

$$(a+b+c)\times(x+y+z) \qquad \text{(Hint: This will not work)}$$

$$(2a+b+c)\times(2x+y+z) \qquad \text{(Does sign change work?)}$$

$$= \mathbf{4ax} + \mathbf{2ay} + \mathbf{2az} +$$

$$\mathbf{2bx} + \mathbf{by} + \mathbf{bz} +$$

$$\mathbf{2cx} + \mathbf{cy} + \mathbf{cz}$$

Observation:

- Sign changes do not help in eliminating terms.
- Coefficients determine term cancellation and adjustment.
- Available terms come from essentials.
- The goal is to adjust coefficients properly using $\frac{3+4}{2}$ to balance.

Fixing Attempt 1

Reminder: We need to separate $\mathbf{bx}, \mathbf{bz}$ and $\mathbf{ay}, \mathbf{cy}$.
Testing with Modified Coefficients:

$$(2a + 2b + c) \times (2x + y + z) \quad : \text{Again, this did not work}$$
$$= \mathbf{4ax} + \mathbf{2ay} + \mathbf{2az} +$$
$$\mathbf{4bx} + \mathbf{2by} + \mathbf{2bz} +$$
$$\mathbf{2cx} + \mathbf{cy} + \mathbf{cz}$$

Observations:

- We still have mixed terms, and separation is not achieved.
- $\mathbf{ay}, \mathbf{bx}$ and $\mathbf{bz}, \mathbf{cy}$ are not properly isolated.
- Further coefficient adjustments are needed.

Fixing Attempt 2

Testing Another Modification:

$$(2a + 2b + c) \times (2x + 2y + z) \quad : \text{Again, this did not work}$$
$$= \mathbf{4ax} + \mathbf{4ay} + \mathbf{2az} +$$
$$\mathbf{4bx} + \mathbf{4by} + \mathbf{2bz} +$$
$$\mathbf{2cx} + \mathbf{2cy} + \mathbf{cz}$$

Observations:

- We still haven't fully separated **ay**, **bx** and **bz**, **cy**.
- More refinement in coefficient selection is needed.

Trying Another Adjustment:

$$(4a + 2b + c) \times (2x + 2y + z) \quad : \text{Again, this did not work}$$
$$= \mathbf{8ax} + \mathbf{8ay} + \mathbf{4az} +$$
$$\mathbf{4bx} + \mathbf{4by} + \mathbf{2bz} +$$
$$\mathbf{2cx} + \mathbf{2cy} + \mathbf{cz}$$

Observations:

- Coefficients are still not balanced.
- The terms **ay**, **bz** and **bz**, **cy** are still not properly separated.
- More refinements are needed to fix the coefficient mismatches.

Fixing 4: (I think it is gonna work)

Balancing the Expression:

$$(4a + 2b + c) \cdot (4x + 2y + z)$$

Expanding:

$$\begin{aligned} = \mathbf{16ax} + \mathbf{8ay} + \mathbf{4az} + \\ \mathbf{8bx} \quad + \mathbf{4by} + \mathbf{2bz} + \\ \mathbf{4cx} \quad + \mathbf{2cy} + \mathbf{cz} \end{aligned}$$

Key Observations:

- Now all coefficients are equally balanced.
- Terms include: $16ax$, cz, and other essential terms.

Matching with Essentials:

- ax (First Essential) is already included via $16ax$.
- cz (Second Essential) is already present.
- Terms az, by, and cx are obtained from:

$$\frac{ThirdEssential + FourthEssential}{2}$$

Remaining Terms:

$$8ay + 8bx + 2bz + 2cy$$

Next Steps:

- Process the remaining terms in the next steps.

Breaking Down the Remaining Terms

Initial Expression:

$$8ay + 8bx + 2bz + 2cy$$

Step 1: Factor and Divide

$$= 4(ay + bx) + bz + cy$$

Step 2: Subtract Essential Terms

$$-16ax - cz$$

Step 3: Remove Essential 4 Contribution

$$-(ay + bx) - bz - cy \quad \text{(From Essential 4)}$$

Step 4: Remaining Terms

$$3(ay + bx)$$

Toom-3 Recurrence Relation

General Form:

Toom-3 multiplication splits two numbers into three parts and evaluates at strategic points. The recurrence relation is:

$$T(n) = 5T(n/3) + O(n)$$

Breakdown:

- The input is divided into three parts.
- Three recursive multiplications occur at different points.
- Interpolation reconstructs the result.

Complexity:

$$T(n) = \Theta(n^{\log_3 5}) \approx O(n^{1.46})$$

- **Breakdown:**

Identifier	Components	Processes	Poly-Order
I_4	ax	As high order	4
I_3	ay + bx	$E_5 - 12E_1 + 3E_2 - 4(I_2)$	3
I_2	az + by + cx	$\frac{E_3+E_4}{2} - E_1 - E_2$	2
I_1	bz + cy	$E_5 - E_3 - 15E_1 - 3(I_2 + I_3)$	1
I_0	cz	As low order	0

- **Formula:**

$$P \times Q = I_4 \cdot x^{\frac{4n}{3}} + I_3 \cdot x^{\frac{3n}{3}} + I_2 \cdot x^{\frac{2n}{3}} + I_1 \cdot x^{\frac{n}{3}} + I_0$$

- **Note:**

$$x\ is\ the\ base\ of\ the\ number.$$

$$In\ binary,\ x\ =\ 2.$$

$$In\ decimal,\ x\ =\ 10.$$

- **Multiplication Methods and Their Complexities:**

Method	Time Complexity	Best Used For
Naïve (Schoolbook)	$O(n^2)$	Small numbers
Karatsuba	$O(n^{1.585})$	Medium-sized numbers
Toom-3	$O(n^{1.464})$	Larger numbers
FFT Multiplication	$O(n \log n)$	Very large numbers

Toom-3 Essentials Extraction (Polynomial Approach)

- **Integer Representation in TOOM-3:**

$$P = [\_\_\_\_\_a\_\_\_\_\_|\_\_\_\_\_b\_\_\_\_\_|\_\_\_\_\_c\_\_\_\_\_] \quad \text{(n bits)}$$

$$Q = [\_\_\_\_\_X\_\_\_\_\_|\_\_\_\_\_y\_\_\_\_\_|\_\_\_\_\_z\_\_\_\_\_] \quad \text{(n bits)}$$

- We express $P(x)$ and $Q(x)$ as polynomials by partitioning their coefficients:

$$P(x) = a \cdot x^2 + b \cdot x + c$$

$$Q(x) = X \cdot x^2 + y \cdot x + z$$

Important Points for Essential Calculations

- $x = 0$
- $x = 1$
- $x = -1$
- $x = 2$
- $x = \infty$

Why These Points Matter:

- $P(0), Q(0)$ isolates the lowest-order coefficient.
- $P(1), Q(1)$ and $P(-1), Q(-1)$ capture sum and alternation of terms.
- $P(2), Q(2)$ emphasizes higher-order terms.
- $P(\infty), Q(\infty)$ extracts the leading coefficient.
- These evaluations allow efficient reconstruction of the product.

$$P(x) = a \cdot x^2 + b \cdot x + c$$
$$Q(x) = X \cdot x^2 + y \cdot x + z$$

1. **At $x = 0$ (Essential 2: Slide 8)**

$$P(0) = c, \quad Q(0) = z$$

$$E_0 = c \cdot z$$

2. **At $x = 1$ (Essential 3: Slide 8)**

$$P(1) = a + b + c, \quad Q(1) = X + y + z$$

$$E_1 = (a + b + c)(X + y + z)$$

3. At $x = -1$ (Essential 4: Slide 10)

$$P(-1) = a - b + c, \quad Q(-1) = X - y + z$$

$$E_{-1} = (a - b + c)(X - y + z)$$

4. At $x = 2$ (Essential 5: Slide 15)

$$P(2) = 4a + 2b + c, \quad Q(2) = 4X + 2y + z$$

$$E_2 = (4a + 2b + c)(4X + 2y + z)$$

5. At $x = \infty$ (Essential 1: Slide 8)

$$P(\infty) \approx a \cdot x, \quad Q(\infty) \approx X \cdot x$$

$$E_\infty = a \cdot X$$

Toom-3 Formula for Calculating Product (Polynomial Approach)

- **Formula**

$$P \times Q = I_4 \cdot x^{\frac{4n}{3}} + I_3 \cdot x^{\frac{3n}{3}} + I_2 \cdot x^{\frac{2n}{3}} + I_1 \cdot x^{\frac{n}{3}} + I_0$$

$$Refer\ to\ Slide\ (20)$$

  - **Note:**

$$0\ to\ \infty\ are\ essential\ 2,\ 3,\ 4,\ 5,\ 1\ respectively.$$

x	$P(x)$	$Q(x)$	E_x
0	c	z	$c \cdot z$
1	$a+b+c$	$X+y+z$	$(a+b+c)(X+y+z)$
-1	$a-b+c$	$X-y+z$	$(a-b+c)(X-y+z)$
2	$4a+2b+c$	$4X+2y+z$	$(4a+2b+c)(4X+2y+z)$
∞	$a \cdot x$	$X \cdot x$	$a \cdot X$

Introduction to Graphs

Definition: A graph is a mathematical structure used to model pairwise relationships between objects. It consists of:

- **Nodes (Vertices):** Represent objects or entities in the graph.
- **Edges:** Represent relationships or connections between two nodes.

Types of Graphs:

- **Directed Graph (Digraph):** Edges have a direction (e.g., one-way streets).
- **Undirected Graph:** Edges do not have a direction (e.g., mutual friendships).
- **Weighted Graph:** Edges have weights representing costs or distances.
- **Unweighted Graph:** All edges are treated equally.

Applications: Used in networks, social media, navigation systems, and many computational problems.

Assignment:

- Construct a graph with n nodes.
- Determine the number of unique paths between nodes.
- Find the minimum number of paths required to travel between any two nodes.

Type: Minimum Spanning Tree (MST)